7th Week

Cryptographic Concepts for Finite Automata

Synopsis.

- One-Way Functions and Hardcores
- Pseudorandom Generators
- Interactive Proof Systems
- Primeimmunity

May 21, 2018. 23:59

Course Schedule: 16 Weeks

Subject to Change

- Week 1: Basic Computation Models
- Week 2: NP-Completeness, Probabilistic and Counting Complexity Classes
- Week 3: Space Complexity and the Linear Space Hypothesis
- Week 4: Relativizations and Hierarchies
- Week 5: Structural Properties by Finite Automata
- Week 6: Stype-2 Computability, Multi-Valued Functions, and State Complexity
- Week 7: Cryptographic Concepts for Finite Automata
- Week 8: Constraint Satisfaction Problems
- Week 9: Combinatorial Optimization Problems
- Week 10: Average-Case Complexity
- Week 11: Basics of Quantum Information
- Week 12: BQP, NQP, Quantum NP, and Quantum Finite Automata
- Week 13: Quantum State Complexity and Advice
- Week 14: Quantum Cryptographic Systems
- Week 15: Quantum Interactive Proofs
- Week 16: Final Evaluation Day (no lecture)

YouTube Videos

- This lecture series is based on numerous papers of T. Yamakami. He gave conference talks (in English) and invited talks (in English), some of which were video-recorded and uploaded to YouTube.
- Use the following keywords to find a playlist of those videos.
- YouTube search keywords:

Tomoyuki Yamakami conference invited talk playlist

Conference talk video

Main References by T. Yamakami

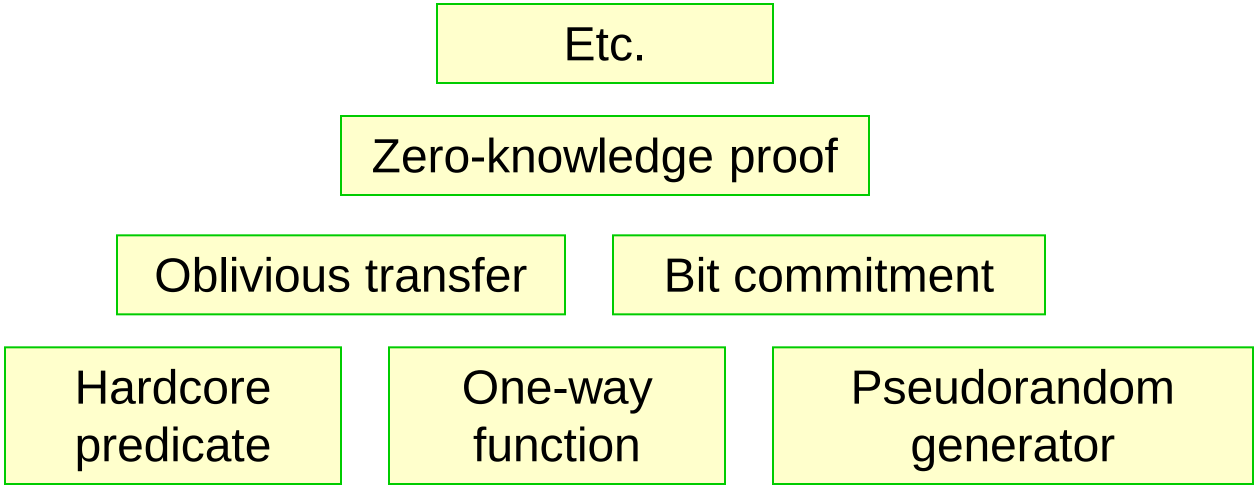
- ✎ T. Yamakami. Immunity and pseudorandomness of context-free languages. Theor. Comput. Sci. 412(45): 6432-6450 (2011)
- ✎ T. Yamakami. Not all multi-valued partial CFL functions are refined by single-valued functions (extended abstract). In Proc. of IFIP TCS 2014, Lecture Notes in Computer Science vol. 8705, pp. 136-150 (2014)
- ✎ T. Yamakami. Structural complexity of multi-valued partial functions computed by nondeterministic pushdown automata. ICTCS 2014, CEUR Workshop Proceedings 1231, CEUR-WS.org 2014, pp. 225-236 (2014)
- ✎ T. Yamakami. Pseudorandom generators against advised context-free languages. Theor. Comput. Sci. 613: 1-27 (2016)

I. One-Way Functions and Pseudorandom Generators

1. Cryptographic Primitives
2. (Strongly) One-Way Functions
3. Weakly One-Way Functions
4. Natural Candidates for OWFs
5. Pseudorandomness
6. Polynomial-Time Indistinguishability
7. Generating Pseudorandom Bits
8. Pseudorandom Generators
9. PEGs Versus OWFs

Cryptographic Primitives

- If we want to build a complex cryptographic system, it is necessary to break it into small building blocks.
- Primitives are such building blocks that support complex cryptographic systems.



What are One-Way Functions?

- Yao (1982) first considered the notion of one-way function.
- Intuitively, a (strongly) one-way function f(x) is
  - Easy to compute from its inputs x, but
  - Hard to invert from its images y=f(x) (i.e., find $x' \in f^{-1}(y)$).

$Prob_{x,A}[f(A(f(x),1^n)) = f(x)] < 1/p(n)$ for any efficient algorithm A, any polynomial p and almost all sizes n.

Input x →  Computes f(x). — Easy!

Input $(y,1^n)$ →  Computes $x \in \{0,1\}^n$ s.t. f(x)=y. — Hard!

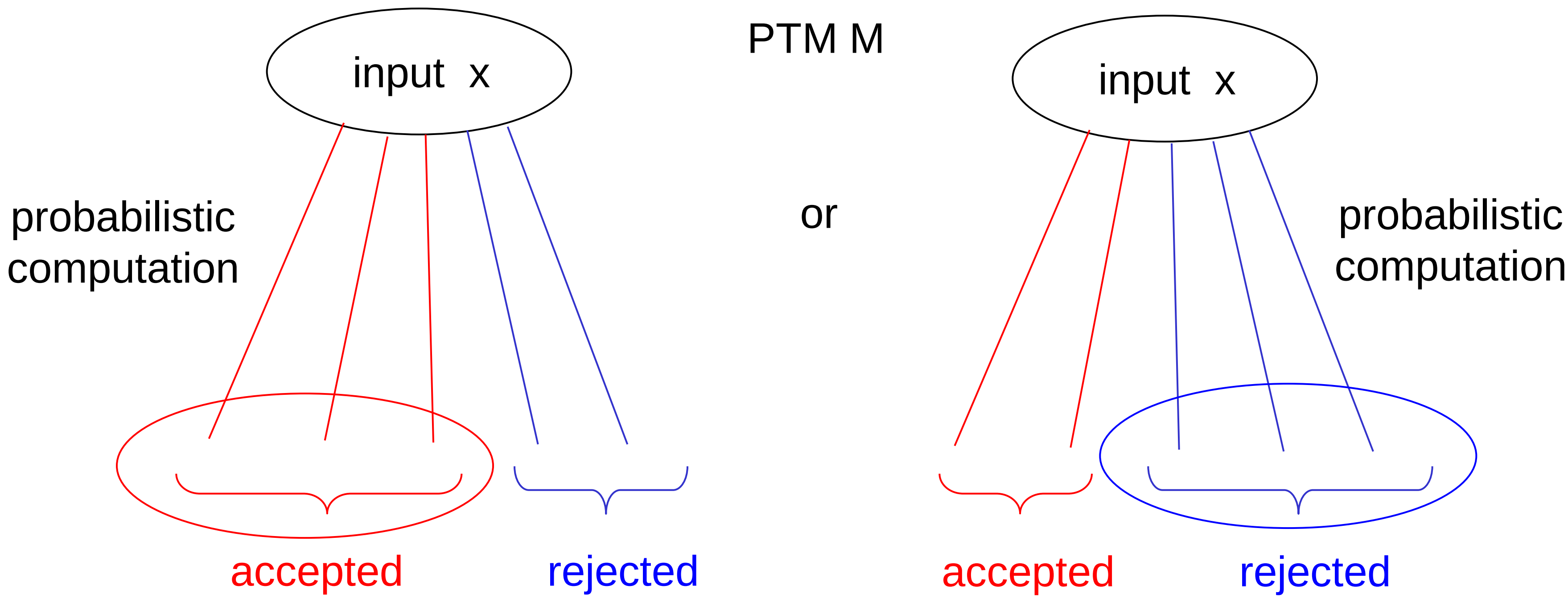
Probabilistic Poly-Time Algorithms (revisited)

- Recall the model of probabilistic Turing machine from Week 2.

- We informally use the term "probabilistic polynomial-time algorithm" to mean "probabilistic polynomial-time Turing machine."

Probabilistic Computation of PTMs (revisited)

- A PTM produces accepting/rejecting computation paths.



$\Pr_M[M(x)=1] > \frac{1}{2}$ M accepts x

$\Pr_M[M(x)=0] \geq \frac{1}{2}$ M rejects x

(Strongly) One-Way Functions I

- Consider a function f : $\{0,1\}^* \rightarrow \{0,1\}^*$.

U_n is a random variable ranging over $\{0,1\}^n$.

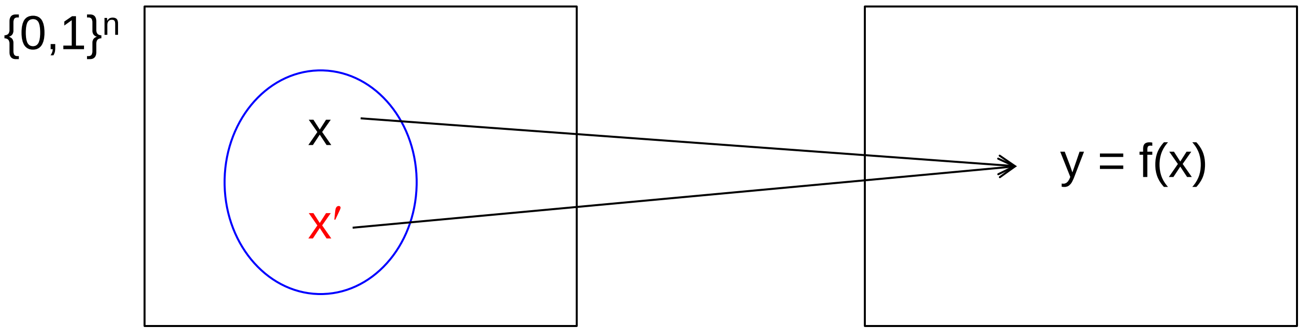
- f is (strongly) one-way if
  1. (easy to compute) there is a deterministic polynomial-time algorithm that computes f, and
  2. (hard to invert) for every probabilistic polynomial-time algorithm A, every positive polynomial p, and for all sufficiently large length n,

$$\Pr_{A,U_n}\left[A(f(U_n),1^n) \in f^{-1}(f(U_n))\right] < \frac{1}{p(n)}$$

(Strongly) One-Way Functions II

$$\Pr\left[A(f(U_n),1^n) \in f^{-1}(f(U_n))\right] < \frac{1}{p(n)}$$

- This formula means:
  - the probability that, on input $(y,1^n)$ with $y \in \{ f(x) \mid x \in \{0,1\}^n \}$, algorithm A finds x′ satisfying f(x′) = y is polynomially small.
- Note that there are possibly many x′ satisfying f(x′) = y.
- So, it suffices to find at least one of them probabilistically.



Weakly One-Way Functions

- There is another notion of one-way function.

- f is weakly one-way if
  1. (easy to compute) there is a deterministic polynomial-time algorithm that computes f, and
  2. (slightly hard to invert) there exists a polynomial p such that, for every probabilistic polynomial-time algorithm A and all sufficiently large length n,

$$\Pr_{A,U_n}\left[A(f(U_n),1^n) \notin f^{-1}(f(U_n))\right] > \frac{1}{p(n)}$$

- (Claim) A strongly one-way function exists $\Leftrightarrow$ a weakly one-way function exists. [Yao (1982)]

Natural Candidates for OWFs I

- Unfortunately, we do not know whether or not one-way functions (OWFs) exist.
- However, we have several good candidates for OWFs.
- The RSA function
  - with index set (N,e), where N is a product of two $(1/2 \cdot \log_2 N)$-bit primes P and Q, and e is an integer smaller than N and relatively prime to (P-1)(Q-1).

$$RSA_{N,e}(x) = x^e \mod N$$

There is no common factor.

- The Rabin function
  - with a similar condition to the above,

$$Rabin_N(x) = x^2 \mod N$$

Natural Candidates for OWFs II

- The DLP (discrete logarithm problem) function
  - with index set (P, G), where P is a $(1/2 \cdot \log_2 N)$-bit prime P and a primitive element G in the multiplicative group modulo P,

$$DLP_{P,G}(x) = G^x \mod P$$

- Open Problems
  - Prove or disprove that the aforementioned candidates are truly one-way functions.
  - More generally, prove or disprove the existence of one-way functions.

Pseudorandomness

- Blum and Micali (1984) considered how to generate a sequence of bits whose next bit is hardly predicted by even powerful adversary.

meaning: "family" or "series"

- In contrast, Yao (1982) considered a sequence that no adversary distinguishes from a uniformly random sequence with a small margin of error.
- Let $X = \{ X_n \}_{n \in N}$ be an ensemble of random variables indexed by N.
- For example, consider an infinite series of fair coins. For each $n \in N$, we define Xn to be the outcome of the flip of the (n+1)th coin.

Polynomial-Time Indistinguishability

- We start with "indistinguishability" of two ensembles of random variables.

- Two ensembles $X = \{ X_n \}_{n \in N}$ and $Y = \{ Y_n \}_{n \in N}$ are indistinguishable in polynomial time (or computationally indistinguishable) if
  - for every probabilistic polynomial-time algorithm M, every positive polynomial p, and all sufficiently large length n,

$$\left| \Pr\left[M(X_n, 1^n) = 1 \right] - \Pr\left[M(Y_n, 1^n) = 1 \right] \right| < \frac{1}{p(n)}$$

The probability of distinguishing between X_n and Y_n is polynomially small.

Generating Pseudorandom Bits

short
truly
random
bits

computer
algorithm

long
pseudorandom
bits

??

truly random
bits

hard to
distinguish
them

true randomness

“uniform” ensemble

$U_{l(n)}$ is chosen uniformly at random.

...udorandom Generators

- An ensemble $X = \{ X_n \}_{n \in N}$ is called pseudorandom if there is a uniform ensemble $U = \{ U_{l(n)} \}_{n \in N}$ such that $\{ G(U_n) \}_{n \in N}$ and U are polynomial-time indistinguishable, where $l: N \to N$ is a fixed function.

- A pseudorandom generator G is a deterministic polynomial-time algorithm satisfying the following two conditions:
  1. (expansion) there is a function $l: N \to N$ (called the expansion/stretch factor of G) such that $l(n) > n$ for all $n \in N$ and $|G(s)| = l(|s|)$ for all $s \in \{0,1\}^*$, and
  2. (pseudorandomness) the ensemble $\{ G(U_n) \}_{n \in N}$ is pseudorandom.

PRGs Versus OWFs

- Let G: $\{0,1\}^* \rightarrow \{0,1\}^*$ be a function with expansion factor $l(n) = 2n$ (that is, $|G(x)| = 2|x|$ for all $x \in \{0,1\}^*$).
- We define a function f: $\{0,1\}^* \rightarrow \{0,1\}^*$ by

$$f(x, y) = G(x)$$

- (Claim) If G is a pseudorandom generator, then f is a strongly one-way function.

- Moreover, we can prove the following.
- (Claim) If there exists a one-way function, then a pseudorandom generator exists. [Håstad-Impagliazzo-Levin-Luby (1999)]

II. Hardcore Functions

1. What are Hardcore Functions?
2. Hardcore Predicates (or Functions)
3. Examples of Hardcores
4. Why are Hardcores so Useful?

What are Hardcore Functions?

- A hardcore function P for a function f is
  - Easy to compute from its inputs x, but
  - Hard to "predict" P(x) from the images f(x) of the function f without knowing inputs x.

$|Prob_{x,A}[A(f(x),1^n) = P(x)] - 1/2^{l(n)}| < 1/p(n)$ for any polynomial p and almost all sizes n.

where l(n) is the size function of P

Input x →  Computes P(x). — Easy!

Input $(y,1^n)$ →  Computes P(x) s.t. $x \in \{0,1\}^n$ and f(x)=y. — Hard!

Hardcore Predicates (or Functions)

- Let b: {0,1}*$\rightarrow$ {0,1} be a polynomial-time computable predicate (i.e., functions outputting 0 or 1).
- Let f: {0,1}*$\rightarrow$ {0,1}* be a function.

- b is a hardcore predicate (or a hardcore) of f if, for every probabilistic polynomial-time algorithm A, every positive polynomial p, and all sufficiently large n,

$$\Pr\left[A(f(U_n) = b(U_n)\right] < \frac{1}{2} + \frac{1}{p(n)}$$

- This means that, to predict the value b(s) from input f(s) is similar to choosing 0 or 1 at random.
- Hardcores actually exist for any strongly one-way function (assuming that one-way functions exist).

Examples of Hardcores

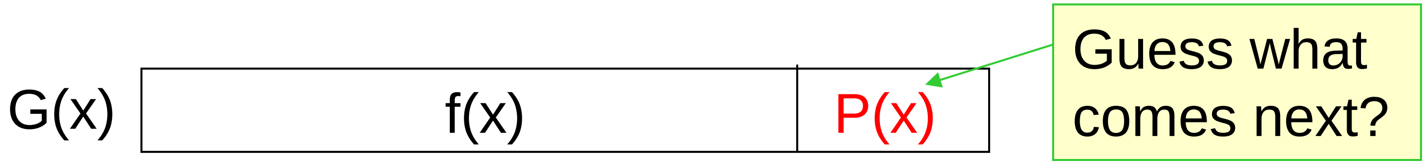
- There are known hardcore predicates for (strongly) one-way functions of a special form (explained below).
- Let b: $\{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ be the (bitwise) inner-product-mod-2 function; that is, $b(x,r) = x \odot r \pmod 2$.
- Example: $$b(1011,1101) = 1 \bullet 1 + 0 \bullet 1 + 1 \bullet 0 + 1 \bullet 1 \pmod 2$$ $$= 2 \pmod 2 = 0$$
- (Claim) Let f be any strongly one-way function. Define g as g(x,r) = f(x)r (concatenation), where |x|=|r|. The predicate b (defined above) is a hardcore of g. [Goldreich-Levin (1989)]

Why are Hardcores so Useful?

- Let f be any one-way permutation and
  let P be any hardcore predicate for f.

- Define G(x) = f(x)P(x) (string concatenation).

- The definition of a hardcore says that we cannot predict the value P(x) from the value f(x) with high confidence.



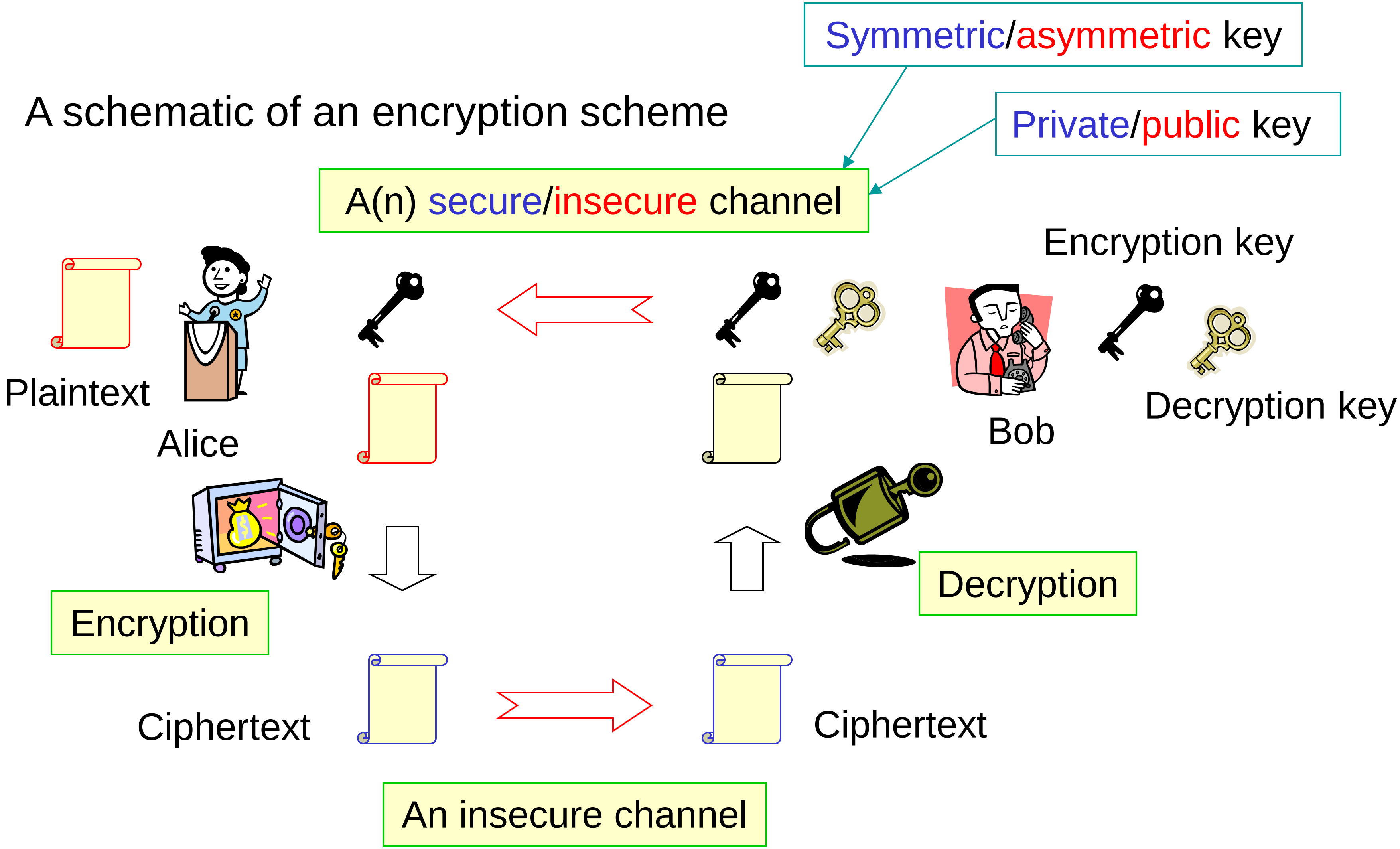
- Well-Known Result: unpredictability = pseudorandomness

- Therefore, this function G(.) is a pseudorandom generator that stretches n bit seeds to n+1 bit strings.

III. Basic Cryptosystems

1. Public-Key Cryptosystems
2. Non-Interactive Bit Commitment

Private-Key/Public-Key Encryption Schemes



Non-Interactive Bit Commitment

- In a non-interactive bit commitment scheme, a committer (Alice) and a verifier (Bob) communicate with each other and satisfy the following conditions.
  - ➢ (hiding) In the commit phase, Alice commits to a single bit b and sends some information z to Bob so that Bob cannot recover b from z,
  - ➢ (binding) In the opening (or reveal) phase, Alice reveals her bit b and Bob checks if b is the correct committed bit from z. We require that Alice cannot cheat Bob by revealing a different bit.

committing (z)

opening (b)

(b)

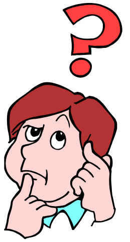
verifier

committer

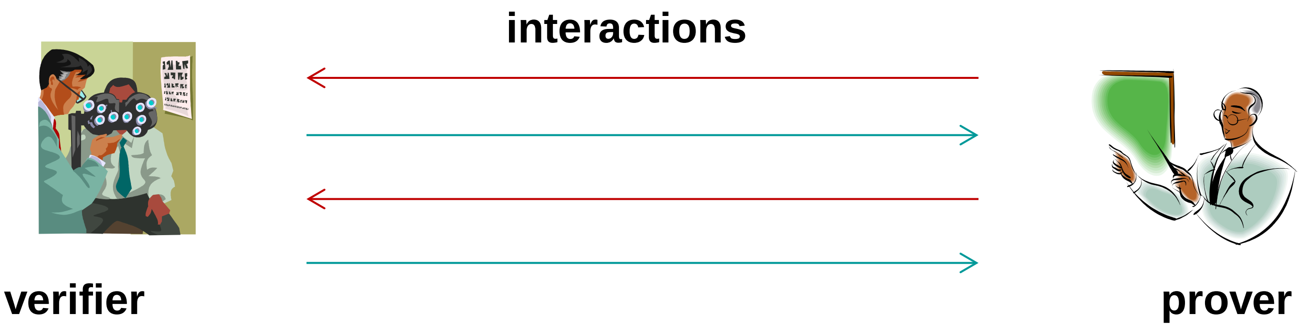
IV. Interactive Proof Systems

1. What is an Interactive Proof?
2. Interactive Proof Systems
3. Constant-Space Interactive Proofs
4. Private Coins vs. Public Coins
5. One-Way Functions for 1-Tape Machines

What is an Interactive Proof?



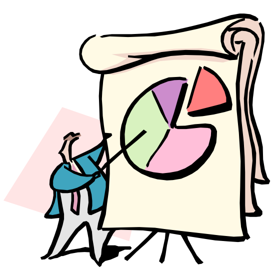
- An interaction between two (or more) parties has been studied in many cryptographic contexts.
- Goldwasser, Micali, and Rackoff (1989) studied a series of interactions between a prover (who presents a proof) and a verifier (who verifies the proof).
- This gave rise to a notion of interactive proof (IP) systems.
- In an IP system, a prover P sends a proof (either correct or wrong) and a verifier V checks if the proof is indeed correct.



Intuitive Definition

- A language L has an IP system ⇔ there exists a verifier V that satisfies the following two conditions:
  1. For every $x \in L$, there exists a honest prover P such that V accepts a proof from P with probability at least 2/3; and
  2. For every $x \notin L$, V rejects any proof from any (possibly malicious) prover with probability at least 2/3.

A proof is a piece of information.



prover

$x \in L$ → Accepts with probability ≥ 2/3

$x \notin L$ → Rejects with probability ≥ 2/3

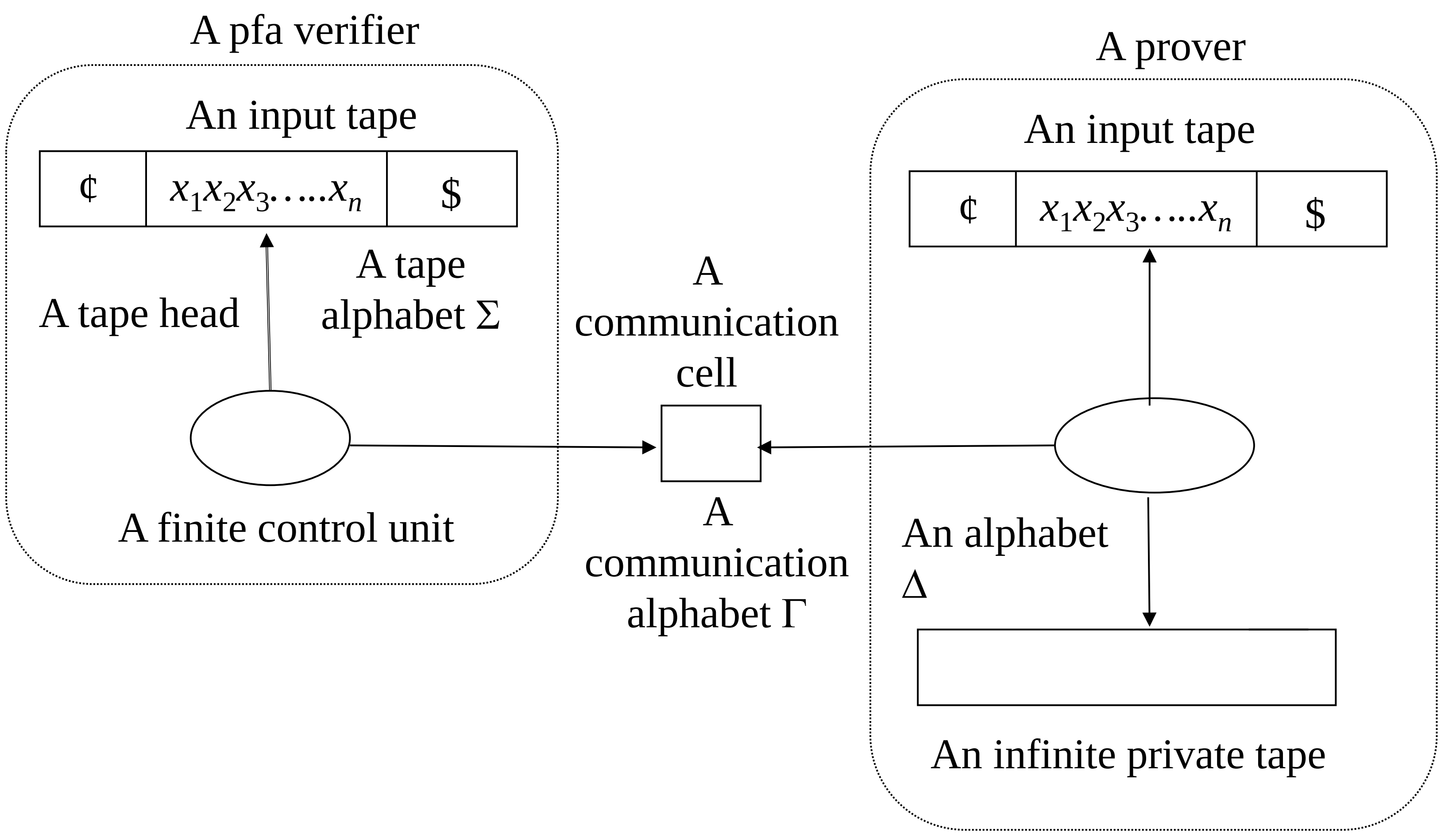
verifier

Believe me. This is a correct proof.

Let me judge the correctness of your proof.

Underlying Machine Model

- Dwork-Stockmeyer IP system is illustrated as follows.



Interactive Proof Systems

- Let (P,V) be a pair of prover P and verifier V.
- Let L be a language over alphabet {0,1}.
- (P,V) is an interactive proof system for L if
  - V is a specified probabilistic machine,
  - (P,V) satisfies the following conditions:

1. (completeness) for every $x \in L$,

$$\Pr\left[(P,V)(x)=1\right] \geq \frac{2}{3}$$

2. (soundness) for any $x \notin L$ and any prover B,

$$\Pr\left[(B,V)(x)=1\right] \leq \frac{1}{3}$$

Constant-Space Interactive Proofs

- Dwork and Stockmeyer (1992) considered interactive proof (IP) systems with 2-way probabilistic finite automata (2pfa's).
- Major advantages: we can prove certain separation results that are impossible (at least at present) to obtain for polynomial-time or logarithmic-space bounded IP systems.
- IP(⟨restrictions⟩) = the class of all languages that have IP systems satisfying the restrictions given in ⟨restrictions⟩.
- For example:
  - IP(2pfa,poly-time) = the class of all languages that have IP systems with 2pfa verifiers running in expected polynomial time.

Private Coins vs. Public Coins

- In an IP system, a verifier obtains random bits (by flipping coins) and decides his next actions. The verifier keeps those random bits secretly. A prover has no way knowing those bits of the verifier.
- This situation is described as the verifier playing with "private coins."
- In contrast, if the verifier reveals his random bits to the prover every time, then this situation is described as the verifier playing with "public coins."

- If the verifier uses "public coins" instead of "private coins," then we write AM(⟨restriction⟩) in place of IP(⟨restriction⟩).

"AM" stands for "Arthur-Merlin game."

Known Results

- Dwork and Stockmeyer (1992) obtained the following results.
- (Claim)
  1. 2PFA $\subseteq$ AM(2pfa) $\subseteq$ IP(2pfa,poly-time) $\subseteq$ IP(2pfa)
  2. Pal = $\{ x \in \{0,1\}^* \mid x = x^R \}$ is in IP(2pfa) but not in AM(2pfa).
  3. Center = $\{ u1v \mid u,v \in \{0,1\}^*, |u|=|v| \}$ is in AM(2pfa) but not in 2PFA.

- (*) We will return to this topic in Week 13.

Track Notation (revisited)

- To describe the notion of one-way function in the 1-tape linear-time model, we need to introduce a “track” notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\cdots\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \text{ where } x = x_1 x_2 \cdots x_n \text{ and } y = y_1 y_2 \cdots y_n$$

- Even if $|x| \neq |y|$, we want to use the same notation to express

$$\begin{bmatrix} x \\ y\#^k \end{bmatrix} \qquad \begin{bmatrix} x\#^k \\ y \end{bmatrix}$$

if $|x| = |y|+k$ and $k \geq 1$ and $|x|+k = |y|$ and $k \geq 1$, respectively, where # is a distinct “blank” symbol.

One-Way Functions for 1-Tape Machines I

- A total function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ is called one-way if
  1. $f \in$ 1-FLIN, and
  2. there is no function $g \in$ 1-FLIN such that

$$f\left(g\left(\begin{bmatrix} f(x) \\ 1^{|x|} \end{bmatrix}\right)\right) = f(x)$$

  for all inputs x.

- When f is length-preserving, the above equality can be replaced by $f(g(f(x))) = f(x)$.

$\forall x \in \Sigma_1^*$ [$|f(x)| = |x|$]

- Theorem: [Tadaki-Yamakami-Lin (2010)]
  - There is no one-way function in 1-FLIN.

- (*) In the next slide, we will see a proof sketch.

One-Way Functions for 1-Tape Machines II

- Recall 1-DLIN and 1-FLIN from Week 1, and 1-FLIN(partial) and 1-NLINMV from Week 6.

❑ Proof Sketch:

- Assume by contradiction that a one-way function f : $\Sigma_1^* \rightarrow \Sigma_2^*$ exists in 1-FLIN.
- Define $f^{-1}([y\ 1^n]^T) = \{ x\#^{|y|-n} \mid |x|=n, f(x) = y \}$ if $|y| \geq n$; $f^{-1}([y\ 1^n]^T) = \{ x \mid |x|=n, f(x) = y \}$ otherwise.
- Clearly, $f^{-1} \in$ 1-NLINMV.
- As seen in Week 6, since 1-NLINMV $\sqsubseteq_{ref}$ 1-FLIN(partial), there is a refinement, say, g of f^{-1} in 1-FLIN(partial).
- We then construct a 1DTM computing g in O(n) time.
- Since $f^{-1} \sqsubseteq_{ref}$ g, M converts f, a contradiction against our assumption.

QED

V. Pseudorandomness for Automata

1. Negligible Functions
2. C-Pseudorandomness
3. Examples of C-Pseudorandom Languages

Negligible Functions

- We apply pseudorandomness to finite automata.

$R^{\geq 0} = \{ z \in R \mid z \geq 0 \}$

- First, we need a notion of negligible function.

- A real-valued function $h: N \to R^{\geq 0}$ is negligible $\Leftrightarrow$
  - $\forall p$: positive polynomial, $h(n) \leq 1/p(n)$ holds for all but finitely many numbers $n \in N$ (super-polynomially small).

- Example: $h(n) = 1/2^n$, $h'(n) = 1/n^{\log(n)}$

Intuition: Pseudorandomness

- $A\Delta L$ denotes the symmetric difference $(A - L)\cup(L - A)$.

- Intuitively, the C-pseudorandomness of L means:
  for any language $A\in C$ and for almost all n's,
  $|(A\Delta L)\cap\Sigma^n|$ is "nearly" a half of $|\Sigma^n|$. (Fig.1)
- Equivalently: for any language $A\in C$ and for almost all n's,
  $|A\cap(L\cap\Sigma^n)|$ is "nearly" equal to $|A\cap(\Sigma^n\text{-}L)|$. (Fig.2)

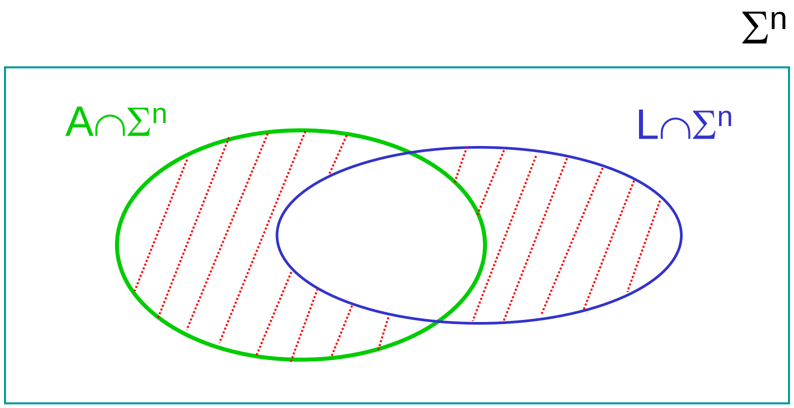


Fig.1

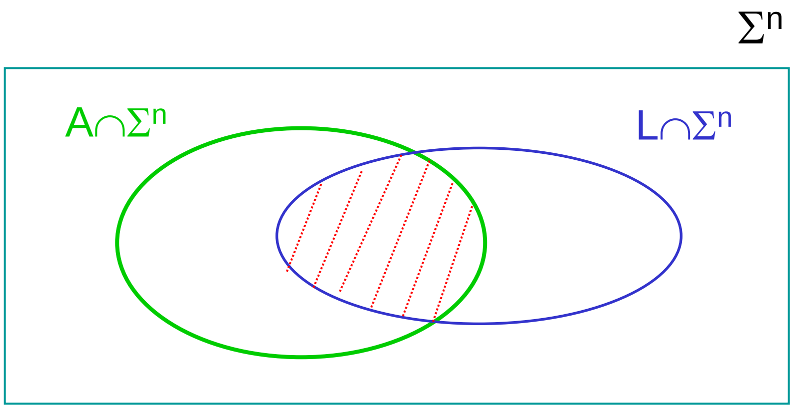


Fig.2

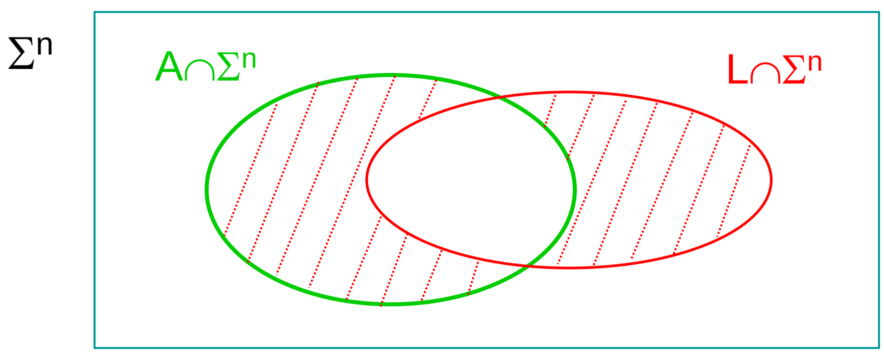
C-Pseudorandomness I

- Let L be any language over Σ with $|\Sigma| \geq 2$.
- Let C be any language family.

- L is C-pseudorandom $\Leftrightarrow$ for all $A \in C$ over Σ,

$$h(n) = \left| \frac{\left|(A\Delta L) \cap \Sigma^n\right|}{\left|\Sigma^n\right|} - \frac{1}{2} \right|$$ is negligible.

- (Claim) No language in C is C-pseudorandom.



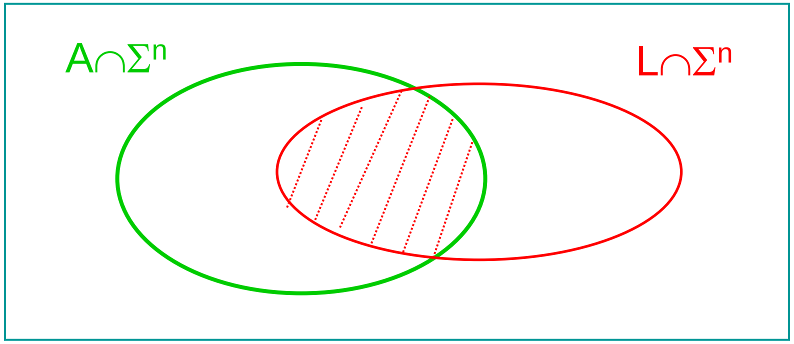
$$\frac{\left|(A\Delta L) \cap \Sigma^n\right|}{\left|\Sigma^n\right|} \to \frac{1}{2}$$

C-Pseudorandomness II

- We may be focused on p-dense languages.
- A language L (over Σ) is weakly C-pseudorandom $\Leftrightarrow$
  - for all p-dense $A \in C$ (over Σ),

    $h'(n) =_{def} \left| \, |(A \cap L) \cap \Sigma^n| \,/\, |A \cap \Sigma^n| - ½ \, \right|$ is negligible.
- A language family D is (weakly) C-pseudorandom $\Leftrightarrow$
  - D contains a (weakly) C-pseudorandom language.
- NOTE: Not known whether NP is P-pseudorandom.

Σ^n



$$\frac{\left|(A \cap L) \cap \Sigma^n\right|}{\left|A \cap \Sigma^n\right|} \rightarrow \frac{1}{2}$$

Examples of C-Pseudorandom Languages

- Let $x \odot y$ denote the (bitwise) binary inner product.
- Consider the following extended language in CFL.

$IP^* = \{\, axy \mid a \in \{\lambda, 0, 1\},\ x, y \in \{0,1\}^*, |x| = |y|, x^R \odot y \equiv 1 \pmod 2 \,\}$

- IP^* is REG/n-pseudorandom. Hence, we obtain:

- Theorem: [Yamakami (2011)]

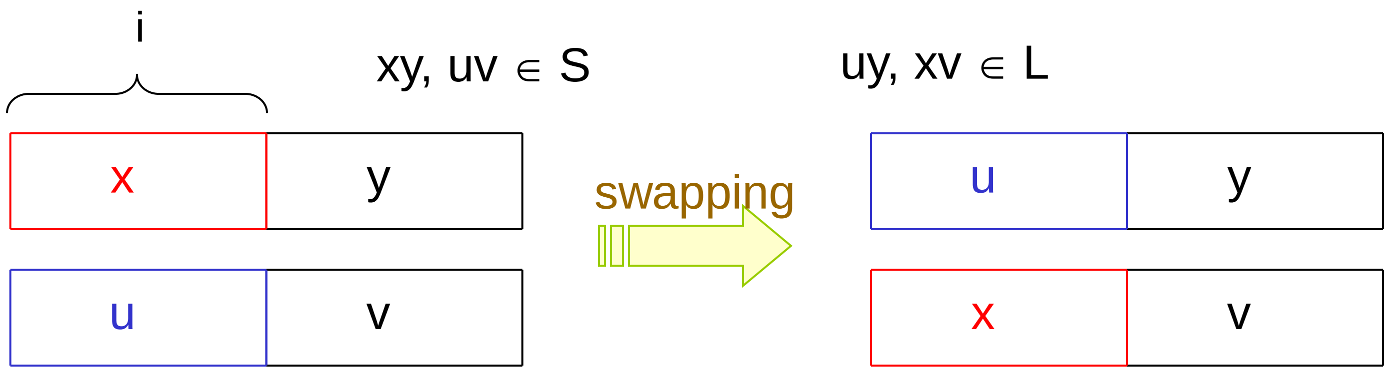
  CFL is REG/n-pseudorandom.

- The proof of this theorem utilizes the swapping lemma for regular languages, discussed in Week 5. (See the next slide.)

Swapping Lemma for REGs (revisited)

Swapping Lemma for REGs [Yamakami (2008),(2010)]

- If L is regular, then $\exists m>0$ s.t. $\forall n\in N$ $\forall S\subseteq L\cap\Sigma^n$ $(|S|\geq m)$ $\forall i\in[n]$ $\exists xy,uv\in S$ $(|x|=|u|=i)$ $[\ xy\neq uv\ \&\ uy,xv\in L\ ]$.



- See Week 5 for the references.

CFL/n-Pseudorandom Languages I

- We discuss CFL/n-pseudorandom languages.
- Consider the languages
  - $IP^{+} = \Sigma^{\leq 8} \cup (IP_3 \cap \Sigma^{\geq 8}) \Sigma^2$, where
  - $IP_3 = \{ axyz \mid a \in \{\lambda,0,1\}, x,y,z \in \{0,1\}^*, |x|=|z|, |y|=2|x|, (xz) \odot y^R \equiv 1 \pmod 2 \}$ (extension of IP*)

- CFL(2)/n is an advised version of CFL(2), which was discussed in Week 5.

- Lemma: [Yamakami (2016)]

  $L \in CFL(2)/n \Leftrightarrow \exists L_1, L_2 \in CFL/n$ s.t. $L = L_1 \cap L_2$.

CFL/n-Pseudorandom Languages II

- Theorem: [Yamakami (2016)]
  1. IP_3 and IP^+ are in $L \cap CFL(2)/n$.
  2. IP_3 and IP^+ are CFL/n-pseudorandom.

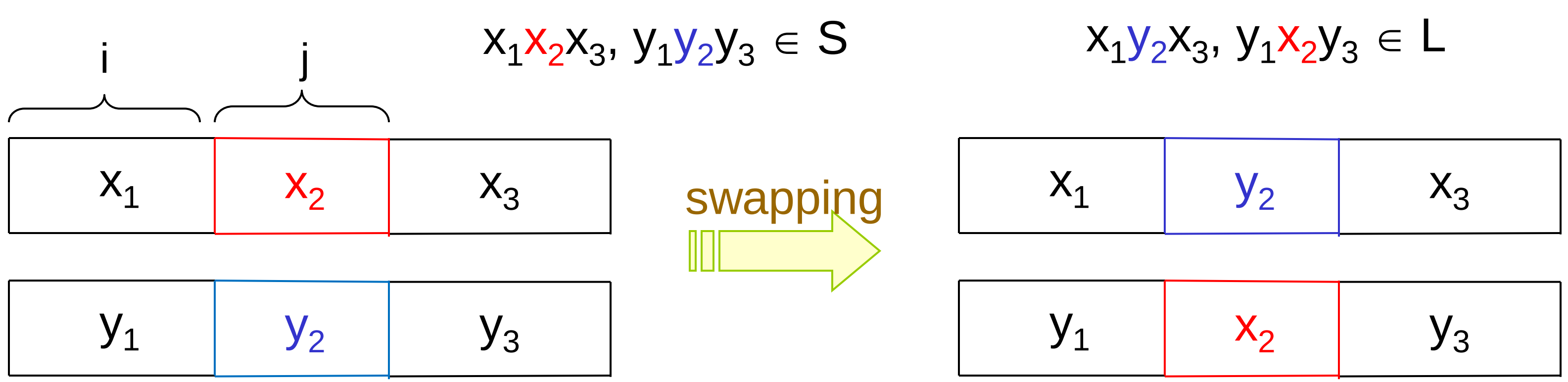
- For the latter claim of the above theorem, we need the swapping lemma for context-free languages discussed in Week 5. (See the next slide.)

- Corollary: [Yamakami (2016)]
  1. $L \cap CFL(2)/n \not\subseteq CFL/n$.
  2. $CFL(2) \not\subseteq CFL/n$.

Swapping Lemma for CFLs (revisited)

Swapping Lemma for CFLs [Yamakami, (2008,2016)]

- If L is context-free, then $\exists m>0$ s.t. $\forall n\geq 2\ \forall S\subseteq L\cap\Sigma^n\ \forall j_0,k_0 \in[2,n-1]_Z(k_0\geq 2j_0)\forall i\in[0,n]\forall j\in[j_0,k_0](i+j\leq n)\forall u\in\Sigma^{j0}$ $(|S_{i,u}|<|S|/m(k_0-j_0+1)(n-j_0+1))\exists x=x_1x_2x_3,y=y_1y_2y_3\in S$ $(|x_1|=|y_1|=i)(|x_2|=|y_2|=j)(|x_3|=|y_3|)$ [$x_2\neq y_2 \,\&\, x_1y_2x_3,y_1x_2y_3\in L$].



- See Week 5 for the references.

Open Problems

- There are many open questions to solve.

1. Is there any CFL/n-pseudorandom language in CFL(2) (instead of CFL(2)/n)?
2. Find natural languages that are C-pseudorandom against D for reasonable language families C and D.

VI. P-Denseness and Primeimmunity

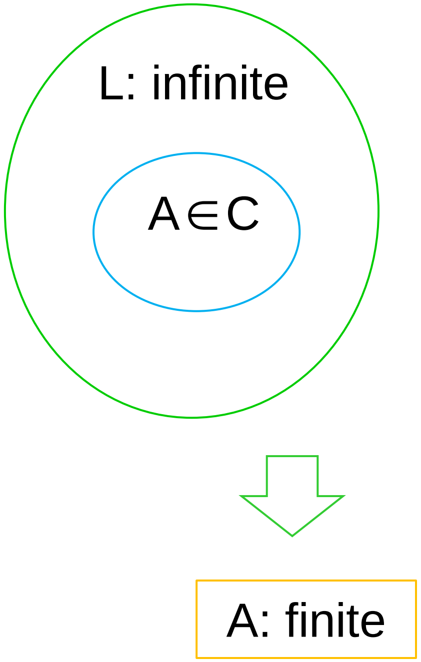
1. P-Denseness
2. P-Dense REG-Immunity
3. C-Primeimmunity
4. Examples of C-Primeimmune Languages
5. C-Bi-Primeimmunity
6. Examples of C-Bi-Primeimmune Languages
7. A Connection to C-Pseudorandomness

C-Immunity (revisited)

- Recall the definition of C-immune languages in Week 5.
- Immunity is concerned with "finiteness."

- Let C be any nonempty language family.

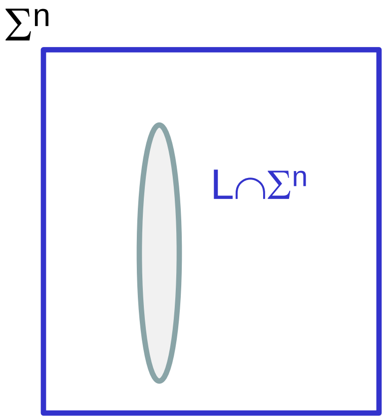
- A language L is C-immune $\Leftrightarrow$
  1) L is infinite, and
  2) no infinite subset A of L exists in C.

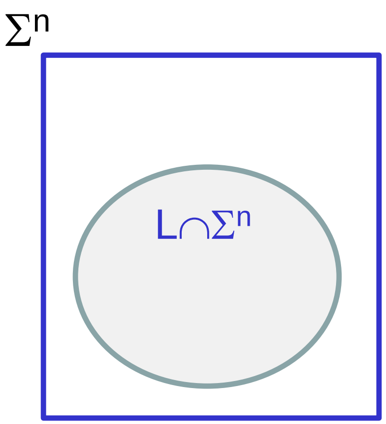


- A language family D is C-immune $\Leftrightarrow$
  - D contains a C-immune language.

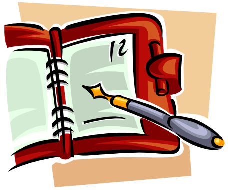
P-Denseness

- All known context-free REG-immune languages L make the ratio $|L \cap \Sigma^n| / |\Sigma^n|$ exponentially small.
  - E.g., L_{eq} and $Pal_{\#}$
- A language L is polynomially dense (or p-dense) $\Leftrightarrow$
  - There is a non-zero polynomial p s.t. $|L \cap \Sigma^n| / |\Sigma^n| \geq 1/p(n)$ for all but finitely many n (i.e., only polynomially small).
- Polynomial denseness is a key to our further discussion.





P-Dense REG-Immunity



- What language family is p-dense REG-immune?
- Theorem: [Yamakami (2011)]

  L $\cap$ CFL/n is p-dense REG-immune.

❑ Proof Sketch:

- Consider the language

  LCenter = $\{ ax0^m10^my \mid a \in \{\lambda, 0, 1\},\ 2^m \leq |x| = |y| < 2^{m+1} \}$.

- Cleraly, LCenter $\in$ L $\cap$ CFL/n. Thus, it suffices to prove
  - ➢ LCenter is p-dense REG-immune,

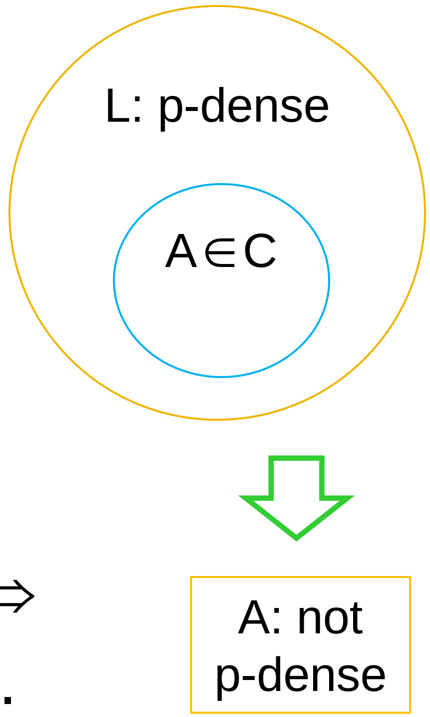
  by the pumping lemma for REGs.

QED

- (Open Problem) Is CFL p-dense REG-immune?

C-Primeimmunity

- Let us introduce a variant of C-immunity using “p-dense” sets in place of “finite” sets.



- Let C be any language family.
- A language L is C-primeimmune $\Leftrightarrow$
  1) L is p-dense, and
  2) L has no p-dense subset in C.
- A language family D is C-primeimmune $\Leftrightarrow$
  - D contains a C-primeimmune language.
- NOTE: p-dense REG-immune $\Rightarrow$ REG-primeimmune

Examples of C-Primeimmune Languages

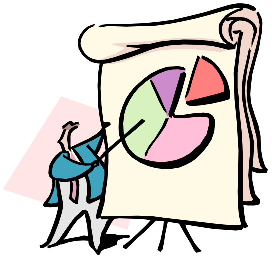
- Equal = { x ∈ {0,1}* | $\#_0(x) = \#_1(x)$ } is not p-dense.
- Here, we consider its extended language:
  - Equal_* = { aw | a ∈ {λ,0,1}, w ∈ Equal }

- (Claim)
  1. Equal_* is p-dense.
  2. Equal_* is in CFL.
  3. Equal_* is not REG-immune.
  4. Equal_* is REG/n-primeimmune.

- Theorem: [Yamakami (2011)]

  CFL is REG/n-primeimmune.

❑ Proof: This comes from Claims 2 & 4 above.

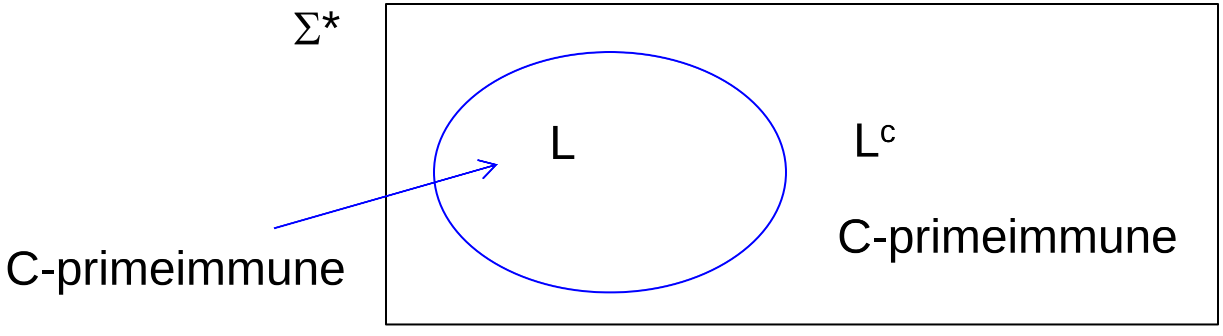


C-Bi-Primeimmunity

- Let C be any language family.

- A language L is C-bi-primeimmune ⇔
  - L and L^c are both C-primeimmune.

- A language family D is C-bi-primeimmune ⇔
  - D contains a C-bi-primeimmune language.



Examples of C-Bi-Primeimmune Languages

- Recall that $x \odot y$ is the (bitwise) inner product of x and y.
- Consider the following language:
  - IP_* = { axy | $a \in \{\lambda,0,1\}$, $x,y \in \{0,1\}^*$, $|x|=|y|$, $x^R \odot y \equiv 1$ (mod 2) }.

- Lemma: [Yamakami (2011)]

  IP_* is REG/n-bi-primeimmune.

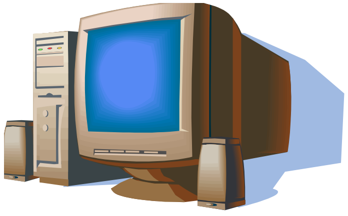
- Since $IP_* \in$ CFL, we conclude the following statement.

- Theorem: [Yamakami (2011)]

  CFL is REG/n-bi-primeimmune.

A Connection to C-Pseudorandomeness

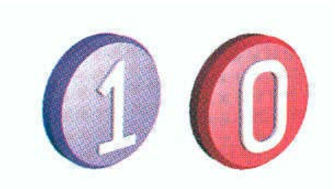
- There is a connection to C-pseudorandomness.
- Lemma: [Yamakami (2011)]
  If L is weakly C-pseudorandom, then it is C-bi-primeimmune.
- The converse does not hold, because the language $Equal_*$ ($\in$ CFL) is REG-primeimmune but not weakly REG-pseudorandom.



VII. PRGs by Finite Automata

1. Pseudorandom Generators
2. Existence and Limitation
3. Proof Idea for the Theorem
4. Generators Against CFL/n

Pseudorandom Generators I



- Let G: {0,1}*→ {0,1}* be any function.

- G has stretch factor s(n) ⇔
  - |G(x)|=s(|x|) for all x ∈ {0,1}*.

- G fools a language A (over {0,1}*) ⇔
  - $l(n) =_{def} | Prob_x[A(G(x))=1] – Prob_y[A(y)=1] |$ is negligible, where |x|=n and |y|=s(|x|).

- Intuitively: A cannot tell the difference between truly random strings y and generated strings G(x).

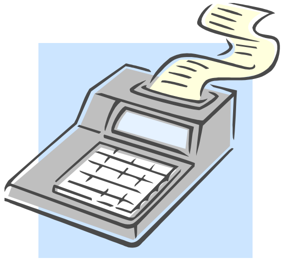
Pseudorandom Generators II

- Let G: {0,1}*→ {0,1}* be any function.

- G is a pseudorandom generator against C ⇔
  - for all A ∈ C (over {0,1}), G fools A.

- G is a weakly pseudorandom generator against C ⇔
  - for all p-dense A ∈ C (over {0,1}), G fools A.

- NOTE: pseudorandom generator ⇒ weakly pseudorandom generator



Connection to Pseudorandom Languages I

- There is a close connection between C-pseudorandom generators and C-pseudorandom languages.
- First, we introduce a notion of almost one-to-oneness.
- Let G: $\{0,1\}^* \rightarrow \{0,1\}^*$ have stretch factor n+1.
- G is almost 1-1 $\Leftrightarrow$
  - There is a negligible function t such that
    $|\{ G(x) \mid x \in \{0,1\}^n \}| = |\{0,1\}^n|(1 - t(n))$ holds for all n.
- NOTE: If G is exactly 1-1, then t(n)=0.

Connection to Pseudorandom Languages II

- Let G: $\{0,1\}^* \to \{0,1\}^*$ be any almost 1-1 function with stretch factor n+1.
- Let $S_G = \{ G(x) \mid x \in \{0,1\}^* \}$ be the image of G.

- Lemma: [Yamakami (2011)]

  G is a (weakly) pseudorandom generator against C $\Leftrightarrow$

  - the image S_G of G is (weakly) C-pseudorandom.

- (Open Problem)

  Can we weaken the above conditions of “almost 1-1” and “stretch factor n+1”?

Existence I

- Here, we show the existence of pseudorandom generators against REG/n.
- Recall the function class $CFLSV_t$.

- Theorem: [Yamakami (2011)]

  There exists an almost 1-1 pseudorandom generator G in $CFLSV_t$ with stretch factor n+1 against REG/n.

- (*) In the next slide, we will give a sketch of the proof of the above theorem.

Existence II

- Proof Sketch:

- First, we define an almost 1-1 function G: $\{0,1\}^* \to \{0,1\}^*$ with stretch factor n+1 such that G $\in$CFLSV$_t$ and S_G = IP_*, where S_G is the image { G(x) | x $\in$ $\{0,1\}^*$ } of G.
- We already know that IP_* is REG/n-pseudorandom.
- Since S_G = IP_*, S_G is REG/n-pseudorandom.
- As seen before, this implies that G is a pseudorandom generator against REG/n.

QED

Non-Existence I

Next, we show a limitation of pseudorandom generators against REG/n.

- Theorem: [Yamakami (2011)]

  There is no almost 1-1 weakly pseudorandom generator in 1-FLIN with stretch factor n+1 against REG.

- (*) In the next slide, we will give a sketch of the proof.

Non-Existence II

❑ Proof Sketch:

- Assume that such a generator G exists.
- Define $H(xb) = G(x)$ for any $b \in \{0,1\}$.
- Since $H \in$ 1-FLIN, it follows that $H^{-1} \in$ 1-NLINMV.
- Take a refinement f of H^{-1} in 1-FLIN(partial) by Week 6.
- Consider the image S_G of G. Note that $y \in S_G \leftrightarrow f(y)\downarrow$.
- Since $f \in$ 1-FLIN(partial), we obtain $S_G \in$ 1-DLIN = REG.
- It follows that S_G is REG-pseudorandom.
- Since REG cannot be weakly REG-pseudorandom, a contradiction follows.

QED

Function Class CFLMV(2)/n

- Before moving to the next subject, we discuss an advised function class, called CFLMV(2)/n.
- Recall CFLMV(2) (= CFLMV $\wedge$ CFLMV) from Week 6.
- Here, we consider its advised version, denoted by CFLMV(2)/n.

- Lemma: [Yamakami (2016)]

  For any multi-valued partial function f, $f \in$ CFLMV(2)/n $\Leftrightarrow$ there exist two multi-valued partial functions g,h $\in$ CFLMV/n such that $f(x) = g(x) \cap h(x)$ for any x.

- In other words, CFLMV(2)/n = CFLMV/n $\wedge$ CFLMV/n.

Generators Against CFL/n I

- Next, we consider pseudorandom generators against CFL/n.

- Theorem: [Yamakami (2016)]

  There exists an almost 1-1 pseudorandom generator G in FL $\cap$ CFLMV(2)/n against CFL/n.

- Note that a famous design-theoretic method of Nisan and Wigderson (1994) does not provide a generator in FL $\cap$ CFLMV(2)/n.

- (*) In the next slide, we will show how to define such a G.

Definition of the Desired Generator

❑ Proof Idea:

- We define the desired generator G as follows.
- Let us set the value G(w) with w = axy and |x|=|y|+1 for $a \in \{ \lambda,0,1 \}$ and $x,y \in \{ 0,1 \}^*$.
- If $a \neq \lambda$, set G(aw) = aG(xy).
- Assume $a = \lambda$. Let x = bz for $b \in \{ 0,1 \}$ and k = (|w|-1)/2.

1. If $w = bzy \wedge z^R \odot y \equiv 1 \pmod 2$, set $G(w) = bzyb^c$.
2. If $w = 1zy \wedge z^R \odot y \equiv 0 \pmod 2$, set G(w) = 1zy1.
3. If $w = 0zy \wedge z^R \odot y \equiv 0 \pmod 2$, there are two cases.
    a. If $\exists$ i [$z_{(k-i-1)}$ = 1, set $G(w) = 0zy_*0$, where y_* is obtained from y by flipping only the i-th bit.
    b. Otherwise, G(w) = 1zy1.

QED

Generators Against CFL/n II

- Here, we present an impossibility result.

- Theorem: [Yamakami (2016)]

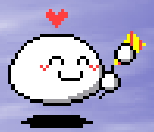
  There is no almost 1-1 weakly pseudorandom generator in CFLMV with stretch factor n+1 against CFL.

- The proof can be done by contradiction.

Open Problems

- There are many open questions to solve.

1. Does a 1-1 PRG against CFL/n exist in CFLMV(2)/n?
2. What happens if we use randomized advice instead of deterministic advice for pseudorandom generators?
3. Is CFL p-sense REG-immune?
4. We can define CFL-primesimple languages. Find CFL-primesimple languages.
5. Is DCFL weakly REG/n-pseudorandom?
6. Construct efficient pseudorandom generators against Σ_k^{CFL}. (See Week 4 for Σ_k^{CFL} .)
7. Find a natural 1-1 pseudorandom generator against REG/n.



Thank you for listening

Q & A

I’m happy to take your question!

END